On the concept of the tunneling time

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Abstract

Asymptotic time evolution of a wave packet describing a non-relativistic particle incident on a potential barrier is considered, using the Wigner phase-space distribution. The distortion of the trasmitted wave packet is determined by two time-like parameters, given by the energy derivative of the complex transmission amplitude. The result is consistent with various definitions of the tunneling time (e.g. the Büttiker-Landauer time, the complex time and Wigner's phase time). The speed-up effect and the negative dispersion are discussed, and new experimental implications are considered.

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Quantum tunneling, where a particle has a chance to pass through classically forbidden regions, was one of the first important applications of wave mechanics. The total barrier penetration probability may be calculated directly from the stationary Schrödinger equation, yet the process time dependence is a more delicate phenomenon. The source of the problem is the uncertainty relation: if the incident momentum was known exactly, the coordinate would be absolutely uncertain, and one could hardly ask a question about the particle transport. In principle, the solution is evident[1]: a wave packet must be prepared beyond the potential domain, far from it, so that one can afford a wide spread, compatible with a relatively small momentum uncertainty. After that, one has to wait long enough until the wave packet would penetrate through the barrier and leave it completely, being splitted in two, forward and backward. However, the problem needs an intricate analysis based upon the time-dependent formalism.

The question: "How much time does the tunneling process take?", has been standing since 1932 (e.g. Ref.[2]). Various definitions, approaches, experiments and reviews were worked out (a partial list is given in Refs.[3]-[22]) in order to answer that question which is substantial for physical applications and for a proper understanding of quantum theory.

There are three cardinal approaches to the concept of the tunneling time.

- Path integrals and the semiclassical approximation suggest[9, 10, 20] employing a *complex* time τ^{C}.

- A physical *clock* may be itroduced[23], at least at the level of a "Gedankenexperiment". One way to measure the time spent under the barrier is to study the particle spin precession in an external magnetic field[4, 5], which leads to a Larmor time τ^{L}. Another way is to consider a vibrating barrier [6, 18], getting the Büttiker – Landauer time τ^{BL}.

- One can try to trace the behaviour of the wave packet in interaction with the barrier [2, 3, 17, 19, 22]. This approach has two frail points [6, 7, 21]: i) a dependence on the initial state preparation, ii) spreading of the wave packet outside the potential region.

It was found subsequently [9, 13, 21] that the definitions of τ^{BL}, τ^{L}, and τ^{C} are not inconsistent. On the other hand, for central potential scattering, which is described in terms of partial-wave phase shifts, Wigner[24] (see also in Ref.[1], Ch. 8) introduced a *phase time* τ^{W} (Eq. (1) below) and noted its relation to the *causality principle*. Recent experiments with optical analogs of quantum tunneling[14]-[16] indicate an importance of the phase time for barrier penetration.

The purpose of this work is to investigate general features of deformation of wave packets in the process of tunneling through potential barriers. It is assumed that the initial momentum uncertainty is small, and we look at the large-time asymptotics. The result is expressed in terms of the (complex) transmission amplitude $A(\kappa)$, which is obtained by the solution of the stationary Schrödinger equation with the

energy $\epsilon = \kappa^2/2m$, the particle mass being m. The *total* transmission probability is given by $|A(\kappa)|^2$, and the change of shape of the coordinate probability distribution is determined by two time-like parameters related to the energy derivatives of the transmission amplitude,

$$\tau^{\mathrm{W}} \equiv d(\arg A)/d\epsilon, \quad \tau^{\mathrm{A}} \equiv d(\ln|A|)/d\epsilon, \tag{1}$$

where $d\epsilon = v d\kappa$, and $v = \kappa/m$. (We set $\hbar = 1$ throughout the paper.) It is noteworthy that these parameters appear in other formulae for the tunneling time, namely, $\tau^{\mathrm{C}} = \tau^{\mathrm{W}} - i\tau^{\mathrm{A}}$ and $\tau^{\mathrm{BL}} = |\tau^{\mathrm{C}}|$. The Wigner phase time τ^{W} has the same form as in the central scattering, where $|A| \equiv 1$, in contrast to the problem concerned.

We shall use the Wigner phase-space quasi-distribution, i.e. the Weyl symbol of the density matrix (see e.g. a review in Ref.[25]). There are two reasons for that: i) this formalism enables one to treat both pure and mixed initial states, and quite general types of the wave packets; ii) it is easier to get rid of irrelevant oscillations of the wave functions.

The time evolution of the initial Wigner function $\rho_0(q,p)$ can be given by means of the phase-space evolution kernel[26], which represents the fundamental solution of the Landau – von Neumann equation for the density matrix. Namely, for any time t the Wigner function is

$$\rho_t(q,p) = \int L_t(q,p;q_0,p_0)\rho_0(q_0,p_0)dq_0dp_0. \tag{2}$$

We shall consider the scattering problem, where the initial state is prepared with an average momentum P_0 and has a relatively small momentum dispersion $\Delta p_0 \ll |P_0|$, which are defined as usual,

$$\begin{aligned} P_0 &= \int p\rho_0(q,p)dqdp, \\ (\Delta p_0)^2 &= \int (p - P_0)^2\rho_0(q,p)dqdp. \end{aligned} \tag{3}$$

Similar definitions hold for the central coordinate Q_0 and the dispersion Δq_0. As follows from the uncertainty relation, $\Delta q_0 \Delta p_0 \geq \frac{1}{2}$, and the inequality may be saturated for a set of pure (coherent) states. It is assumed that the potential barrier $V(q) \geq 0$ is located near the origin and has a finite range D, vanishing for $|q| > D$. The initial state must be prepared in the free space, which means that $|Q_0| - \Delta q > D$. The results of the scattering are observed after the wave packets gets out of the potential region, say, when $t > 2|Q_0|m/P_0$.

In the large time asymptotics, the phase-space evolution kernel for the barrier penetration was found[27] to be a sum of two parts, describing transmission and reflection,

$$L_t(q,p;q_0,p_0) \asymp \delta(p - p_0)T(r_+,p_0) + \delta(p + p_0)R(r_-,p_0), \tag{4}$$

where $r_\pm = q_0 + tp_0/m \mp q$. (If the barrier would be absent, one has $T = \delta(r_+)$, $R \equiv 0$, and L_t is just the solution of the classical Poisson equation for free motion.) The

functions T and R have been expressed in terms of integrals involving the transition and reflection coefficients, $A(\kappa)$ and $B(\kappa)$, respectively. In particular, the integral representation for the transmission propagator is

$$T(r,p) = \int_{-\infty}^{\infty} d\sigma e^{-i\sigma r} A(p + \frac{1}{2}\sigma)\overline{A(p - \frac{1}{2}\sigma)}. \tag{5}$$

This representation is manifestly causal. Note that $A(\kappa)$ has the following general properties [28]: it is analytical in the upper half of the complex plane (having poles at Im $\kappa < 0$), $\overline{A(\kappa)} = A(-\bar{\kappa})$, and $\lim_{\kappa\to\infty} A(\kappa) = 1$. Thus the integral in (5) can be considered as a contour integral in the complex σ-plane. If $q > q_0 + vt$, i.e. q would be in advance of the free motion coordinate, the integration contour can be moved up to infinity, so $T(r,p) = 0$ for $r < 0$. Thus no point of the Wigner distribution is transported faster than it would be in the absence of the potential barrier.

The information transport is somewhat smeared by the fact that the phase space distribution is never too local because of the uncertainty relation. Let us consider the observable consequences of the causality arguments presented above. We shall suppose that the final coordinates are measured, and the detector does not discriminate between different momenta. The probability of finding the transmitted particle at q, for asymptotically large t, is given by the following integral,

$$\mathcal{P}_t(q) = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} d\sigma e^{-i\sigma r} A(p + \frac{1}{2}\sigma) A(-p + \frac{1}{2}\sigma)\rho_0(q - vt + r, p), \tag{6}$$

where $v = p/m$. Dealing with this representation, one can make use of the specific features of ρ_0, which is sharp in p and broad in q. Therefore it is reasonable to expand $A(\pm p + \frac{1}{2}\sigma)$ in powers of $(p - P_0)$. On the other hand, it was found[27] that $T(r,p)$ is an exponentially decreasing function of r, so the fact that the r-dependence of ρ_0 is slow may be taken into account. A straightforward calculation shows that one can expand the integral in powers of $\Delta p_0/P_0$. To the first order, the result is

$$\mathcal{P}_t(q) \approx |A(P_0)|^2 \left[\mathcal{P}_t^0(q) + v_0\tau^{\mathrm{W}}\partial\mathcal{P}_t^0/\partial q + v_0\tau^{\mathrm{A}}2M_t(q)\right]. \tag{7}$$

Here

$$\mathcal{P}^0(q) \equiv \int dp\rho_0(q - vt, p), \quad M_t(q) \equiv \int dp(p - P_0)\rho_0(q - vt, p), \tag{8}$$

$\mathcal{P}_t^0$ represents free motion of the initial wave packet including the usual spreading and M_t is the first moment of the p-distribution, which also appears in the free motion. The time parameters τ^{A} and τ^{W} in (7) should be calculated by Eq. (1) at $\epsilon = P_0^2/2m$, and $v_0 = P_0/m$. Note that the total transmission probability is

$$\int dq\mathcal{P}_t(q)/\int dq\mathcal{P}_t^0(q) = |A(P_0)|^2, \tag{9}$$

as usual, since the two other terms in (7) are eliminated by the integration. The expansion in powers of $(\Delta p_0/P_0)$ can be performed to all orders leading to a sum over higher derivatives and higher moments of ρ_0 with coefficients which depend on the barrier shape.

Two terms in Eq. (7) describe a distortion of the coordinate propability distribution. It is natural to expect that $\tau^{\mathrm{A}} > 0$, the tunneling probability is increasing with energy, so the corresponding term is responsible for an advance in the distribution maximum. It is the so-called speed-up effect[17]. One can tell that tunneling filters out low-energy components of the wave packet. As to the other term, the sign of the phase time τ^{W} is not definite. (Roughly speaking, it is positive for narrow barriers and negative for wide barriers). The corresponding term represents interference effects, i.e dispersion due to the barrier. This may result in an additional advance (if $\tau^{\mathrm{W}} < 0$), or in a delay (if $\tau^{\mathrm{W}} < 0$).

Sometimes the effect of τ^{W} is dominating. That is the case when $|A(\kappa)|$ is constant, as for the example of central scattering, considered by Wigner[24]. In recent experiments with photon wave packets[14], the transmission probability was almost insensitive to the wavelength within the light frequency band, so that $\tau^{\mathrm{A}} \ll \tau^{\mathrm{W}}$. If these parameters are of the same order of magnitude their relative influence may be determined by the distance of the detector from the barrier. As the distance (i.e. the time t) is increasing the speed-up effect prevails over the phase-time effect, since the slope of the free probability distribution is getting down because of the wave-packet spreading.

Let us consider a simple example where the initial phase-space distribution is Gaussian,

$$\rho_0(q,p) = C \exp\left[-\frac{(p-P_0)^2}{2(\Delta p_0)^2} - \frac{(q-Q_0)^2}{2(\Delta q_0)^2}\right]. \tag{10}$$

Here $C = (2\pi\Delta p_0 \Delta q_0)^{-1}$ is the normalization constant, and $\Delta p_0 \Delta q_0 = \frac{1}{2}$ if the state is pure. The calculations are straightforward now,

$$\begin{aligned} \mathcal{P}^0_t(q) &= \sqrt{2\pi}\Delta q C \exp\left[-\frac{(q-Q)^2}{2(\Delta q)^2}\right], \\ M_t(q) &= t\frac{(q-Q)(\Delta p_0)^2}{m(\Delta q)^2}\mathcal{P}^0_t(q), \\ Q &\equiv Q_0 + tv_0, \quad (\Delta q)^2 \equiv (\Delta q_0)^2 + (t\Delta p_0/m)^2. \end{aligned} \tag{11}$$

In the essential domain, the coordinate distribution given by Eq. (7) is

$$\mathcal{P}_t(q) = |A(P_0)|^2 \mathcal{P}^0_t(q)\left[1 + v_0\tau_0(q-Q)/(\Delta q)^2\right], \tag{12}$$

where $\tau_0 = 2t\tau^{\mathrm{A}}(\Delta p_0)^2/m - \tau^{\mathrm{W}}$, which may change its sign with time. The maximum of the transmitted distribution is shifted in advance of the that for the free propagation by

$$\Delta Q = 2v_0\tau_0 / \left(\sqrt{1+\zeta^2} + 1\right) \tag{13}$$

where $\zeta = 2v_0\tau_0/\Delta q$. In the limit of $\Delta p_0 \to 0$, $\Delta q \to \infty$, one has $\zeta \ll 1$, and $\Delta Q \approx v_0\tau_0$. Besides, for $(\Delta p_0)^2/m \ll \tau^{\mathrm{W}}/\tau^{\mathrm{A}}t$, i.e. if the time is not too large (the detector is not too far from the barrier) we get $\tau_0 \approx -\tau^{\mathrm{W}}$ and $\Delta Q \approx -v_0\tau^{\mathrm{W}}$ in agreement with Wigner's prediction and recent experiments[14]. For large positive

τ_0 (which may happen at large t), the shift of the maximum is bounded by the width of the final distribution. Besides, the transmitted distribution is contracted by interaction with the barrier, an effect which is called negative dispersion.

Equality (7) may be applied to any other distribution, not necessarily Gaussian. An example considered in recent microwave simulations of tunneling[11] is a step function. That distribution cannot be realized in quantum mechanics, but it may be considered as a test function. The shift of the half-height point with respect to the "free propagation" has been also calculated from Eq. (7). The result is $v_0\tau_h$, where $\tau_h = t\tau^{\mathrm{A}}(\Delta p_0)^2/m - \tau^{\mathrm{W}}$. Here the speed-up effect is half of that in the Gaussian case, Eq. (12). This is consistent with the causality arguments; the wave-packet front does not move faster because of a potential barrier. The motion of the peak, as well as the motion of the half-height point, have been measured experimentally[14]-[16].

Qualitatively, nonrelativistic particle movement is similar to propagation of light signal through a reflecting stack, like in the experiment of the Berkeley group[14]. The interpretation of the results based upon the light interference picture leads to qualitatively similar results[29]. In order to make a quantitative prediction, one has to calculate the complex transmission coefficient $A(\kappa)$. If the optical barrier may be prepared with a strong frequency dependence, the effect of the "amplitude time" τ^{A} would be observable, besides the "phase time" τ^{W}. Remarkably, the result of the measurements would depend on the distance between the detector and the barrier region.

It should be emphasized, in conclusion, that the wave packet shape must be taken into the consideration of tunneling through potential barriers. Causality is not violated of course, but it manifests itself indirectly in terms of deformation of the wave packet. The phase-space formalism, introduced by Wigner, is quite appropriate for the investigation, and the main corrections for a wave packet with a fairly definite momentum are given in Eq. (7). The resulting effect is an interplay of those involving a couple of time-like parameters, defined by Eq. (1) and owing to the momentum dependence of the complex transmission amplitude. Thus, the complex time τ^{C} introduced previously[9, 20] appears actually in the final distribution; its real part is coupled to the coordinate derivative of the freely propagating distribution and its imaginary part is coupled to the first moment of the momentum distribution.

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References

[1] M. L. Goldberger and K. Watson, *Collision Theory* (J. Wiley, NewYork-London-Sydney, 1964), Ch. 3.

[2] L. A. MacColl, Phys. Rev. **40**, 621 (1932).

[3] T. E. Hartman, J. Appl. Phys. **33**, 3427 (1962).

[4] A. I. Baz, Yad. Fiz. **4**, 252 (1966), and **5**, 229 (1967) [Sov. J. Nucl. Phys. **4**, 182 (1967), and **5**, 161 (1967)].

[5] V. F. Rybachenko, Yad. Fiz. **5**, 895 (1966) [Sov. J. Nucl. Phys. **5**, 635 (1967)].

[6] M. Büttiker and R. Landauer, Phys. Rev. Lett. **49**, 1739 (1982).

[7] C. R. Leavens and G. C. Aers, Phys. Rev. **B39**, 1202 (1989).

[8] E. H. Hauge and J. A. Støvneng, Rev. Mod. Phys. **61**, 917 (1989).

[9] H.A. Fertig, Phys. Rev. Lett. **65**, 2321 (1990).

[10] W. Jaworski and D. M. Wardlaw, Phys. Rev. A. **43**, 5137 (1991).

[11] A. Ranfagni, D. Mugnai, P. Fabeni, and G. P. Pazzi, Appl. Phys. Lett. **58**, 774 (1991).

[12] V. S. Olkhovsky and E. Recami, Phys. Rep. **214**, 339 (1992).

[13] J. G. Muga, S. Brouard, and R. Sala, J. Phys.: Condensed Matter **4**, L579 (1992).

[14] A. M. Steinberg, P. G. Kwiat, and R. Y. Chiao, Phys. Rev. Lett. **71**, 708 (1993).

[15] A. Enders and G. Nimtz, Phys. Rev. **B47**, 9605 (1993).

[16] Ch. Spielmann, R. Szipocs, A. Sting, and F. Krausz, Phys. Rev. Lett. **73**, 2308 (1994).

[17] R. S. Dumont and T. L. Marchioro II, Phys. Rev. **A47**, 85 (1993).

[18] Th. Martin and R. Landauer, Phys. Rev. **A47**, 2023 (1993).

[19] A. M. Steinberg and R. Y. Chiao, Phys. Rev. **A49**, 3283 (1994).

[20] D. Sokolovski, S. Brouard, and J. N. L. Connor, Phys. Rev. **A50**, 1240 (1994).

[21] R. Landauer and Th. Martin, Rev. Mod. Phys. **66**, 217 (1994).

[22] A. M. Steinberg, Phys. Rev. Lett. **74**, 2405 (1995).

[23] A. Peres, Am. J. Phys. **48**, 552 (1980).

[24] E. P. Wigner, Phys. Rev. **98**, 145 (1955).

[25] M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, Phys. Rep. **106**, 121 (1984).

[26] M. S. Marinov, Phys. Lett. **A153**, 5 (1991).

[27] M. S. Marinov and B. Segev, "Quantum Tunneling in the Wigner Representation", quant-ph/9602017, submitted to Phys. Rev. A.

[28] M. S. Marinov and B. Segev, "Analytical properties of scattering amplitudes in one-dimensional quantum theory", quant-ph/9602015, J. Phys. A (to be published).

[29] Y. Japha and G. Kurizki, Phys. Rev. **A53**, 586 (1996).